UNIT-I

ORDINARY DIFFERENTIAL EQUATIONS

Higher order differential equations with constant coefficients – Method of variation of parameters – Cauchy's and Legendre's linear equations – Simultaneous first order linear equations with constant coefficients.

The study of a differential equation in applied mathematics consists of three phases.

(i) Formation of differential equation from the given physical situation, called modeling.
(ii) Solutions of this differential equation, evaluating the arbitrary constants from the given conditions, and
(iii) Physical interpretation of the solution.

HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS.

General form of a linear differential equation of the nth order with constant coefficients is

$$\frac{d^n y}{dx^n} + K_1 \frac{d^{n-1} y}{dx^{n-1}} + K_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots\dots\dots\dots + K_n y = X \qquad \dots\dots\dots\dots (1)$$

Where $K_1, K_2, \dots\dots\dots K_n$ are constants.

The symbol D stands for the operation of differential

(i.e.,) Dy = $\frac{dy}{dx}$, similarly $D^2 y = \frac{d^{2y}}{dx^2}, D^3 y = \frac{d^3 y}{dx^3}, etc...$

The equation (1) above can be written in the symbolic form

$(D^n + K_1 D^{n-1} + \dots\dots\dots + K_n)y = X$ i.e., f(D)y = X

Where f (D) = $D^n + K_1 D^{n-1} + \dots\dots\dots\dots + K_n$

Note

1. $\frac{1}{D} X = \int X dx$
2. $\frac{1}{D-a} X = e^{ax} \int X e^{-ax} dx$
3. $\frac{1}{D+a} X = e^{-ax} \int X e^{ax} dx$

(i) The general form of the differential equation of second order is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Where P and Q are constants and R is a function of x or constant.

(ii)Differential operators:

The symbol D stands for the operation of differential

(i.e.,) Dy = $\frac{dy}{dx}$, $D^2 y = \frac{d^2y}{dx^2}$

$\frac{1}{D}$ Stands for the operation of integration

$\frac{1}{D^2}$ Stands for the operation of integration twice.

(1) can be written in the operator form

$D^2y + PDy + Qy = R$ (Or) $(D^2 + PD + Q)y = R$

(iv) Complete solution = Complementary function + Particular Integral

PROBLEMS

1. **Solve** $(D^2 - 5D + 6)y = 0$

Solution: Given $(D^2 - 5D + 6)y = 0$

The auxiliary equation is $m^2 - 5m + 6 = 0$`

i.e., m = 2,3

$\therefore C.F = Ae^{2x} + Be^{3x}$

The general solution is given by

$$y = Ae^{2x} + Be^{3x}$$

2. **Solve** $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 3y = 0$

Solution: Given $\left(D^2 - 6D + 3y\right) = 0$

The auxiliary equation is $m^2 - 6m + 13 = 0$

i.e., $m = \frac{6 \pm \sqrt{36 - 52}}{2}$

$= 3 \pm 2i$

Hence the solution is $y = e^{3x}(A\cos 2x + B\sin 2x)$

3. Solve $(D^2 + 1) = 0$ **given y(0) =0, y''(0) = 1**

Solution: Given $(D^2+1) = 0$

A.E is $m^2+1=0$

M = $\pm i$

Y = A cosx + B sinx

Y(x) = A cosx + B sinx

Y(0) = A =0

Y'(0) =B =1

$\therefore$ A = 0, B = 1

i.e., y = (0) cosx + sinx

y = sinx

3. **Solve** $(D^2-4D+13)y=e^{2x}$

Solution: Given $(D^2-4D+13)y=e^{2x}$

The auxiliary equation is $m^2-4m+13=0$

$$m=\frac{4\pm\sqrt{16-52}}{2}=\frac{4\pm\sqrt{-36}}{2}=2\pm 3i$$

$$\therefore C.F=e^{2x}\left(A\cos 3x+B\sin 3x\right)$$

$$\text{P.I.} = \frac{1}{D^2-4D+13}e^{2x}$$

$$=\frac{1}{4-8+13}e^{2x}$$

$$=\frac{1}{9}e^{2x}$$

$\therefore$ y = C.F +P.I.

$$y = e^{2x}\left(A\cos 3x+B\sin 3x\right)+\frac{1}{9}e^{2x}$$

5. Find the Particular integral of y''- 3y' + 2y = e$^{x}-e^{2x}$

Solution: Given y''- 3y' + 2y = e$^{x}-e^{2x}$

$$(D^2-3D+2)y=e^x-e^{2x}$$

$$\text{P.I}_1=\frac{1}{D^2-3D+2}e^x$$

$$=\frac{1}{1-3+4}e^x$$

$$=\frac{1}{0}e^x$$

$$=x\frac{1}{2D-3}e^x$$

$$=x\frac{1}{2-3}e^x$$

$$=-xe^x$$

$\text{P.I}_2 = \frac{1}{D^2 - 3D + 2}\left(-e^{2x}\right)$

$= -\frac{1}{4-6+2}e^{2x}$

$= -x\frac{1}{2D-3}e^{2x}$

$= -x\frac{1}{4-3}e^{2x}$

$= -xe^{2x}$

$\therefore \text{P.I.} = \text{P.I}_1 + P.I_2$

$= -xe^{x} + (-xe^{2x})$

$= -x(e^{x} + e^{2x})$

6. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = -2\cosh x$

Solution: Given $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = -2\cosh x$

The A.E is $m^2 - 4m + 5 = 0$

$m = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$

$\text{C.F} = e^{-2x}\left(A\cos x + B\sin x\right)$

$\text{P.I} = \frac{1}{D^2 + 4D + 5}\left(-2\cosh x\right) = -2\frac{1}{D^2 + 4D + 5}\left[\frac{e^x + e^{-x}}{2}\right]$

$= \frac{-1}{D^2 + 4D + 5}e^{x} + \frac{-1}{D^2 + 4D + 5}e^{-x}$

$= \frac{-e^x}{1+4+5} - \frac{e^x}{1-4+5}$

$= \frac{-e^x}{10} - \frac{e^{-x}}{2}$

$\therefore y = C.F + P.I$

$= e^{-2x}\left(A\cos x + B\sin x\right) - \frac{e^x}{10} - \frac{e^{-x}}{2}$

Problems based on P.I = $\frac{1}{f(D)}\sin ax(or)\frac{1}{f(D)}\cos ax$

$\Rightarrow$ **Replace** $\mathbf{D}^2$ $by -a^2$

7. Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin 3x$

Solution: Given $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=\sin 3x$

The A.E is $m^2+3m+2=0$

$(m+1)(m+2) = 0$

$M = -1, m = -2$

C.F = $Ae^{-x}+Be^{-2x}$

$$P.I = \frac{1}{D^2+3D+2}\sin 3x$$

$$= \frac{1}{-3^2+3D+2}\sin 3x \text{ (Replace } D^2 by - a^2)$$

$$= \frac{1}{3D-7}\sin 3x = \frac{1}{3D-7}\frac{(3D+7)}{(3D+7)}\sin 3x$$

$$= \frac{3D+7}{(3D)^2-(7)^2}\sin 3x$$

$$= \frac{3D+7}{9D^2-49}\sin 3x$$

$$= \frac{3D+7}{9(-3^2)-49}\sin 3x$$

$$= \frac{3D+7}{-130}\sin 3x$$

$$= -\frac{1}{130}\left(3D(\sin 3x)+7\sin 3x\right)$$

$$= -\frac{1}{130}\left(9\cos 3x+7\sin 3x\right)$$

$\therefore y = C.F + P.I$

$$Y = Ae^{-x}+Be^{-2x}-\frac{1}{130}\left(9\cos 3x+7\sin 3x\right)$$

8. Find the P.I of $(D^2+1)=\sin x$

Solution: Given $(D^2+1)=\sin x$

$$P.I. = \frac{1}{D^2+1}\sin x$$

$$= \frac{1}{-1+1}\sin x$$

$$= x\frac{1}{2D}\sin x$$

$$= \frac{x}{2}\frac{1}{D}\sin x$$

$$= \frac{x}{2}\int \sin x dx$$

$$P.I = -\frac{x\cos x}{2}$$

9. Find the particular integral of $(D^2+1)y = \sin 2x \sin x$

Solution: Given $(D^2+1)y = \sin 2x \sin x$

$$=-\frac{1}{2}(\cos 3x - \cos x)$$

$$= -\frac{1}{2}\cos 3x + \frac{1}{2}\cos x$$

$$P.I_1 = \frac{1}{D^2+1}\left[-\frac{1}{2}\cos 3x\right]$$

$$= -\frac{1}{2}\frac{1}{-9+1}\cos 3x$$

$$= \frac{1}{16}\cos 3x$$

$$P.I_2 = \frac{1}{D^2+1}\left[\frac{1}{2}\cos x\right]$$

$$= \frac{1}{2}\frac{1}{-1+1}\cos x$$

$$= \frac{1}{2}x\frac{1}{2D}\cos x$$

$$= \frac{x}{4}\int \cos x dx$$

$$= \frac{x}{4}\sin x$$

$$\therefore P.I = \frac{1}{16}\cos 3x + \frac{x}{4}\sin x$$

Problems based on R.H.S = $e^{ax} + \cos ax (or) e^{ax} + \cos ax$

10. Solve $(D^2 - 4D + 4)y = e^{2x} + \cos 2x$

Solution: Given $(D^2 - 4D + 4)y = e^{2x} + \cos 2x$

The Auxiliary equation is $m^2 - 4m + 4 = 0$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$C.F = (Ax + B)e^{2x}$

$$P.I_1 = \frac{1}{D^2 - 4D + 4}e^{2x}$$

$$= \frac{1}{4-8+4}e^{2x}$$

$$= \frac{1}{0}e^{2x}$$

$$= x\frac{1}{2D-4}e^{2x}$$

$$= x\frac{1}{0}e^{2x}$$

$$= x^2 \frac{1}{2} e^{2x}$$

$$P.I_2 = \frac{1}{D^2 - 4D + 4} \cos 2x$$

$$= \frac{1}{-2^2 - 4D + 4} \cos 2x$$

$$= \frac{1}{-4D} \cos 2x$$

$$= \frac{-1}{4} \left[\frac{1}{D} \cos 2x \right]$$

$$= \frac{-1}{4} \frac{\sin 2x}{2} = -\frac{\sin 2x}{8}$$

$$\therefore y = C.F + P.I$$

$$y = (Ax + B)e^{2x} - \frac{\sin 2x}{8}$$

Problems based on R.H.S = x

Note: The following are important

- $(1+x)^{-1} = 1 - x + x^2 - x^3 + ...$
- $(1-x)^{-1} = 1 + x + x^2 + x^3 + ...$
- $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + ...$
- $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + ...$

11. Find the Particular Integral of $(D^2+1)y = x$

Solution: Given $(D^2+1)y = x$

A.E is $(m^2 - 1) = 0$

$m = \pm 1$

C.F = $Ae^{-x} + Be^{x}$

$$P.I = \frac{1}{D^2 - 1} x$$

$$= \frac{-1}{1 - D^2} x$$

$$= -\left[1 - D^2\right]^{-1} x$$

$$= -\left[1 + D + \left(D^2\right)^2 + ...\right] x$$

$$= -\left[x + 0 + 0 + 0...........\right]$$

$$= -x$$

12. Solve: $\left(D^4 - 2D^3 + D^2\right) y = x^3$

Solution: Given $\left(D^4-2D^3+D^2\right)y=x^3$

The A.E is $m^4-2m^3+m^2=0$

$$m^2(m^2-2m+1)=0$$

$$m^2(m-1)^2=0$$

$$m=0,0\ ,\ m=1,1$$

$$C.F=(A+Bx)e^{0x}+(C+Dx)e^{x}$$

$$P.I=\frac{1}{D^4-2D^3+D^2}x^3$$

$$=\frac{1}{D^2\left[1+\left(D^2-2D\right)\right]}x^3$$

$$=\frac{1}{D^2}\left[1+\left(D^2-2D\right)\right]^{-1}x^3$$

$$=\frac{1}{D^2}\left[1-\left(D^2-2D\right)+\left(D^2-2D\right)^2-\left(D^2-2D\right)^3+..............\right]$$

$$=\frac{1}{D^2}\left[1+2D+3D^2+4D^3+D^4\right]x^3$$

$$=\frac{1}{D^2}\left[x^3+6x^2+18x+24\right]$$

$$=\frac{1}{D}\left[\frac{x^4}{4}+\frac{6x^3}{3}+\frac{18x^3}{2}+24x\right]$$

$$=\frac{x^5}{20}+\frac{6x^4}{12}+\frac{18x^3}{6}+\frac{24x^2}{2}$$

$$=\frac{x^5}{20}+\frac{x^4}{2}+3x^3+12x^2$$

$\therefore y=C.F+P.I$

$$y=(A+Bx)e^{0x}+(C+Dx)e^{x}+\frac{x^5}{20}+\frac{x^4}{2}+3x^3+12x^2$$

Problems based on R.H.S = $e^{ax}x$

P.I = $\frac{1}{f(D)}e^{ax}x=e^{ax}\frac{1}{f(D+a)}x$

13. Obtain the particular integral of ($D^2-2D+5)y=e^x\cos 2x$

Solution: Given ($D^2-2D+5)y=e^x\cos 2x$

$$\textbf{P.I} = \frac{1}{D^2-2D+5}e^x\cos 2x$$

$$= e^x\left[\frac{1}{\left(D\ \ +1\right)^2-2\left(D+1\right)+5}\right]\cos 2x\left(\mathrm{Re}\, place D by D=1\right)$$

$$= e^{x}\left[\frac{1}{D^{2}+2D+1-2D-2+5}\right]\cos 2x$$

$$= e^{x}\left[\frac{1}{D^{2}+4}\right]\cos 2x$$

$$= e^{x}\frac{1}{-4+4}\cos 2x$$

$$= e^{x}x\frac{1}{2D}\cos 2x$$

$$= \frac{xe^{x}}{2}\int \cos 2x dx$$

P.I $= \dfrac{xe^{x}\sin 2x}{4}$

14. Solve ($D+2)^{2}y = e^{-2x}\sin x$

Solution: Given ($D+2)^{2}y = e^{-2x}\sin x$

A.E is (m$^{2}+1) = 0$

m = -2, -2

C.F. = (Ax + B)e^{-2x}

P.I = $\dfrac{1}{(D+2)^{2}}e^{-2x}\sin x$

$= \mathbf{e}^{-2x}\dfrac{1}{(D-2+2)^{2}}\sin x$

$= \mathbf{e}^{-2x}\dfrac{1}{D^{2}}\sin x$

$= \mathbf{e}^{-2x}\dfrac{1}{-1}\sin x$

$= \mathbf{-e}^{-2x}\sin x$

$\therefore y = C.F + P.I$

y = **(Ax + B)e**$^{-2x}$ **- e**$^{-2x}\sin x$

Problems based on f(x) = x$^{n}\sin ax (or) x^{n}\cos ax$

To find P.I when f(x) = x$^{n}\sin ax (or) x^{n}\cos ax$

P.I = $\dfrac{1}{f(D)}x^{n}\sin ax (or) x^{n}\cos ax$

$$\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V + \left[\frac{d}{dD}\frac{1}{f(D)}\right]V$$

i.e., $\dfrac{1}{f(D)}(xV) = x\dfrac{1}{f(D)}V - \left[\dfrac{f^{'}(D)}{f(D)}\dfrac{1}{f(D)}\right]V$

$$\frac{1}{f(D)}xV = x\frac{1}{f(D)}V - \frac{f^{'}(D)}{[f(D)]^{2}}V$$

15. Solve $(D^2+4D+3)y = e^{-x}\sin x + xe^{3x}$

Solution: The auxiliary equation is $m^2+4m+3=0$

m= -1,-3

C.F = $Ae^{-x}+Be^{-3x}$

$$\mathbf{P.I}_1 = \frac{1}{(D^2+4D+3)}\left(e^{-x}\sin x\right)$$

$$= \frac{1}{\left[(D-1)^2+4(D-1)+3\right]}\sin x$$

$$= e^{-x}\frac{1}{\left(D^2+2D\right)}\sin x$$

$$= e^{-x}\frac{(1+2D)}{-1+4D^2}\sin x$$

$$= \frac{e^{-x}}{-5}\left[2\cos x+\sin x\right]$$

$$\mathbf{P.I}_2 = e^{3x}\frac{1}{\left((D+3)^2+4(D+3)+3\right)}x$$

$$= e^{3x}\frac{1}{\left(D^2+10D+24\right)}x$$

$$= \frac{e^{3x}}{24}\left[1+\frac{D^2+10D}{24}\right]^{-1}x$$

$$= \frac{e^{3x}}{24}\left(1-\frac{5D}{12}\right)x$$

$$= \frac{e^{3x}}{24}\left(x-\frac{5}{12}\right)$$

General solution is $y = C.F + P.I$

$$y = Ae^{-x}+Be^{-3x}-\frac{e^{-x}}{5}\left[2\cos x+\sin x\right]+\frac{e^{3x}}{24}\left(x-\frac{5}{12}\right)$$

16 Solve $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y = x\cos x$

Solution: A.E : $m^2+2m+1=0$

m = -1,-1

C.F = $(A+Bx)e^{-x}$

$$\mathbf{P.I} = \frac{1}{(D+1)^2}(x\cos x)$$

$$= \left[x-\frac{2(D+1)}{(D+1)^2}\right]\frac{1}{(D+1)^2}(\cos x)$$

$$= \left[x - \frac{2}{(D+1)}\right]\frac{1}{(D^2+2D+1)}(\cos x)$$

$$= \left[x - \frac{2}{D+1}\right]\frac{1}{(-1+2D+1)}(\cos x)$$

$$= \left[x - \frac{2}{D+1}\right]\frac{\sin x}{2}$$

$$= \frac{x\sin x}{2}$$

$$= \frac{x\sin x}{2} - \frac{\sin x}{D+1}$$

$$= \frac{x\sin x}{2} - \frac{(D-1)\sin x}{D^2-1}$$

$$= \frac{x\sin x}{2} + \frac{\cos x - \sin x}{2}$$

The solution is y = $(A+Bx)e^{-x} + \frac{x\sin x}{2} + \frac{\cos x - \sin x}{2}$

17. Solve $(D^2+1)y = \sin^2 x$

Solution: A.E : m $^2 + 1 = 0$

m $= \pm i$

C.F =A cosx +B sinx

P.I $= \frac{1}{D^2+1}\sin^2 x$

$$= \frac{1}{D^2+1}\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{2}\left\{\frac{1}{D^2+1}e^{0x} - \frac{1}{D^2+1}\cos 2x\right\}$$

$$= \frac{1}{2}\left\{1 + \frac{1}{3}\cos 2x\right\}$$

$$= \frac{1}{2} + \frac{1}{6}\cos 2x$$

$\therefore y =$ A cosx +B sinx $+ \frac{1}{2} + \frac{1}{6}\cos 2x$

18. Solve $\frac{d^2y}{dx^2} - y = x\sin x + (1+x)e^x$

Solution: A.E : $m^2 - 1 = 0$

$m = \pm 1$

C.F = A $e^{-x} + Be^{x}$

$$\text{P.I}_1 = \frac{1}{f(D)}(xV) = \left[x - \frac{f'(D)}{f(D)}\right]\frac{1}{f(D)}(V)$$

$$= \left[x - \frac{2D}{D^2-1}\right]\frac{1}{(D^2-1)}(\sin x)$$

$$= \left[x - \frac{2D}{D^2-1}\right]\frac{\sin x}{-2}$$

$$= \left[-\frac{x\sin x}{2} + \frac{2\cos x}{2(D^2-1)}\right]$$

$$= -\frac{x\sin x}{2} - \frac{\cos x}{2}$$

$$\text{P.I}_2 = \frac{1}{D^2-1}(1+x^2)e^x$$

$$= e^x\frac{1}{[(D+1)^2-1]}(1+x^2)$$

$$= \text{e}^x\frac{1}{(D^2+2D)}(1+x^2)$$

$$= \frac{e^x(2x^3-3x^2+9x)}{12}$$

$$\therefore \text{y} = \quad \text{A}e^{-x} + Be^x + \frac{e^x(2x^3-3x^2+9x)}{12}$$

19. Solve $\frac{d^2y}{dx^2} - y = xe^x \sin x$

Solution: A.E : $\text{m}^2 - 1 = 0$

$$\text{m} = \pm 1$$

$$\text{C.F} = \text{A}e^{-x} + Be^x$$

$$\text{P.I} = \frac{1}{(D^2-1)}xe^x \sin x$$

$$= \text{e}^x\frac{1}{[(D+1)^2-1]}(x\sin x)$$

$$= \text{e}^x\frac{1}{(D^2+2D)}(x\sin x)$$

$$= \text{e}^x\left[x\frac{1}{(D^2+2D)}\sin x - \frac{2D+2}{(2D-1)^2}\sin x\right]$$

$\sin \alpha x = \sin x,\ \alpha = 1, put D^2 = -\alpha^2 = -1$

$$= \text{e}^x\left[x\frac{1}{2D-1}\sin x - \frac{(2D+2)}{(2D-1)^2}\sin x\right]$$

$$= \text{e}^x\left[-x\frac{(1+2D)}{(1-4D^2)}\sin x - \frac{(2D+2)\sin x}{(4D^2-4D+1)}\right]$$

Put $\text{D}^2 = -1$

$$= \text{e}^x\left[-x\frac{(1+2D)}{5}\sin x + \left[\frac{(2D+2)(3-4D)}{9-16D^2}\right]\sin x\right]$$

$$= e^x\left[\frac{-x}{5}[\sin x + 2\cos x] + \left(\frac{-8D^2 - 2D + 6}{9 - 16D^2}\right)\sin x\right]$$

$$= e^x\left[\frac{-x}{5}(\sin x + 2\cos x) + \frac{(14 - 2D)}{25}\sin x\right]$$

$$\text{P.I} = e^x\left[\frac{-x}{5}(\sin x + 2\cos x) + \frac{(14\sin x - 2\cos x)}{25}\right]$$

Complete Solution is

$$y = Ae^{-x} + Be^x + e^x\left[\frac{-x}{5}(\sin x + 2\cos x) + \frac{(14\sin x - 2\cos x)}{25}\right]$$

METHOD OF VARIATION OF PARAMETERS

This method is very useful in finding the general solution of the second order equation.

$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = X$ [Where X is a function of x] ……………….(1)

The complementary function of (1)

$$\text{C.F} = c_1 f_1 + c_2 f_2$$

Where c_1, c_2 are constants and f_1 and f_2 are functions of x

Then P.I $= Pf_1 + Qf_2$

$$P = -\int \frac{f_2 X}{f_1 f_2^1 - f_1^{'} f_2}\,dx$$

$$Q = \int \frac{f_1 X}{f_1 f_2^{'} - f_1^{'} f_2}\,dx$$

$$\therefore y = c_1 f_1 + c_2 f_2 + P.I$$

1. Solve $(D^2 + 4)y = \sec 2x$

Solution: The A.E is $m^2 + 4 = 0$

$$m = \pm 2i$$

$$\text{C.F} = C_1\cos 2x + C_2\sin 2x$$

$f_1 = \cos 2x$ $\quad$ $f_2 = \sin 2x$

$f_1^{'} = -2\sin 2x$ $\quad$ $f_2^{'} = 2\cos 2x$

$$f_1 f_2^{'} - f_1^{'} f_2 = 2\cos^2 2x + 2\sin^2 2x$$

$$= 2[\cos^2 2x + \sin^2 2x]$$

$$= 2[1]$$

$$= 2$$

$$P = -\int \frac{f_2 X}{f_1 f_2^1 - f_1^{'} f_2}\,dx$$

$$= -\int \frac{\sin 2x \sec 2x}{2} dx$$

$$= -\frac{1}{2}\int \sin 2x \frac{1}{\cos 2x} dx$$

$$= \frac{1}{4}\int \frac{-2\sin 2x}{\cos 2x} dx$$

$$= \frac{1}{4}\log(\cos 2x)$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos 2x \sec 2x}{2} dx$$

$$= \frac{1}{2}\int \cos 2x \frac{1}{\cos 2x} dx$$

$$= \frac{1}{2}\int dx$$

$$= \frac{1}{2}x$$

$$\text{P.I} = Pf_1 + Qf_2$$

$$= \frac{1}{4}\log(\cos 2x)\,(\cos 2x) + \frac{1}{2}x\sin 2x$$

2. Solve by the method of variation of parameters $\frac{d^2y}{dx^2} + y = x\sin x$

Solution: The A.E is $m^2 + 1 = 0$

$$m = \pm i$$

$$\text{C.F} = C_1\cos x + C_2\sin x$$

Here $f_1 = \cos x$ $\quad f_2 = \sin x$

$f_1' = -\sin x$ $\quad f_2' = \cos x$

$$f_1 f_2' - f_1' f_2 = \cos^2 x + \sin^2 x = 1$$

$$P = -\int \frac{f_2 X}{f_1 f_2^1 - f_1' f_2} dx$$

$$= -\int \frac{\sin x(x\sin x)}{1} dx$$

$$= -\int x\sin^2 x dx$$

$$= -\int x\frac{(1-\cos 2x)}{2} dx$$

$$= -\frac{1}{2}\int (x - x\cos 2x) dx$$

$$= -\frac{1}{2}\int x dx + \frac{1}{2}\int x\cos 2x dx$$

$$= -\frac{1}{2}\left[\frac{x^2}{2}\right]+\frac{1}{2}\left[x\left(\frac{\sin x}{2}\right)-(1)\left(\frac{-\cos 2x}{4}\right)\right]$$

$$= -\frac{x^2}{4}+\frac{x}{4}\sin 2x+\frac{1}{8}\cos 2x$$

$$Q = \int \frac{f_1 X}{f_1 f_2^{'} - f_1^{'} f_2} dx$$

$$= \int \frac{(\cos x)x(\sin x)}{1} dx$$

$$= \int x \sin x \cos x dx$$

$$= \int x \frac{\sin 2x}{2} dx$$

$$= \frac{1}{2}\int x \sin 2x dx$$

$$= \frac{1}{2}\left[x\left(\frac{-\cos 2x}{2}\right)-(1)\left(\frac{-\sin 2x}{4}\right)\right]$$

$$= -\frac{x}{4}\cos 2x+\frac{1}{8}\sin 2x$$

$$\text{P.I} = \text{Pf}_1 + Qf_2$$

$$= \left[\frac{-x^2}{4}+\frac{x}{4}\sin 2x+\frac{1}{8}\cos 2x\right]\cos x+\left[\frac{-x}{4}\cos 2x+\frac{1}{8}\sin 2x\right]\sin x$$

4. **Solve $(D^2 - 4D + 4)y = e^{2x}$ by the method of variation of parameters**.

Solution: The A.E is $m^2 - 4m + 4 = 0$

$$(m-2)^2 = 0$$

$$m = 2,2$$

$$\text{C.F} = (Ax+B)e^{2x}$$

$$= Axe^{2x} + Be^{2x}$$

$$f_1 = xe^{2x} \qquad f_2 = e^{2x}$$

$$f_1^{'} = xe^{2x}2 + e^{2x}, \; f_2^{'} = 2e^{2x}$$

$$f_1 f_2^{'} - f_2 f_1^{'} = 2x\left(e^{2x}\right)^2 - e^{2x}\left(2xe^{2x} + e^{2x}\right)$$

$$= 2\text{x}\left(e^{2x}\right)^2 - 2x\left(e^{2x}\right)^2 - \left(e^{2x}\right)^2$$

$$= \left(e^{2x}\right)^2[2x - 2x - 1]$$

$$= -\left(e^{2x}\right)^2$$

$$= -e^{4x}$$

$$\text{P} = -\int \frac{f_2 X}{f_1 f_2^1 - f_1^{'} f_2} dx$$

$$= -\int \frac{e^{2x}e^{2x}}{e^{-4x}} dx$$

$$= \int dx = x$$

$$Q = \int \frac{f_1 X}{f_1 f_2^{'} - f_1^{'} f_2} dx$$

$$= \int \frac{xe^{2x} e^{2x}}{-e^{4x}} dx$$

$$= \int -x dx$$

$$= -\frac{x^2}{2}$$

$$\text{P.I} = x^2 e^{2x} - \frac{x^2}{2} e^{2x} = \frac{x^2}{2} e^{2x}$$

y = C.F + P.I

$$= (Ax + B)\, e^{2x} + \frac{x^2}{2} e^{2x}$$

5. Use the method of variation of parameters to solve $(D^2+1)y = \sec x$

Solution: Given $(D^2+1)y = \sec x$

The A.E is $m^2 + 1 = 0$

$m = \pm i$

$$\text{C.F} = c_1 \cos x + c_2 \sin x$$

$$= c_1 f_1 + c_2 f_2$$

$$f_1 = \cos x, f_2 = \sin x$$

$$f_1^{'} = -\sin x, f_2^{'} \cos x$$

$$f_1 f_2^{'} - f_1^{'} f_2 = \cos^2 x + \sin^2 x = 1$$

$$P = -\int \frac{f_2 X}{f_1 f_2^{1} - f_1^{'} f_2} dx$$

$$= -\int \frac{\sin x \sec x}{1} dx$$

$$= -\int \frac{\sin x}{\cos x} dx$$

$$= -\int \tan x dx$$

= log (cos x)

$$Q = \int \frac{f_1 X}{f_1 f_2^{'} - f_1^{'} f_2} dx$$

$$= \int \frac{\cos x \sec x}{1} dx$$

$$= \int dx$$

= x

$$\text{P.I} = Pf_1 + Qf_2$$

$$= \log(\cos x)\cos x + x \sin x$$

$$y = c_1 \cos x + c_2 \sin x + \log(\cos x)\cos x + x \sin x$$

6. **Solve** $\left(D^2+a^2\right)y=\tan ax$ **by the method of variation of parameters**.

Solution: Given $\left(D^2+a^2\right)y=\tan ax$

The A.E is $m^2+a^2=0$

$m=\pm ai$

C.F = $c_1\cos ax+c_2\sin ax$

$f_1=\cos ax, f_2=\sin ax$

$f_1^{'}=-a\sin x, f_2^{'}=a\cos ax$

$$f_1f_2^{'}-f_2f_1^{'}=a\cos ax\cos ax-\sin ax(-a\sin ax)$$
$$=a\cos^2 ax+a\sin^2 ax$$
$$=a(\cos^2 ax+\sin^2 ax)$$
$$=\text{a}$$

P.I = Pf_1+Qf_2

$$\text{P}=-\int\frac{f_2X}{f_1f_2^1-f_1^{'}f_2}dx$$
$$=-\int\frac{\sin ax\tan ax}{a}dx$$
$$=-\frac{1}{a}\int\frac{\sin^2 ax}{\cos ax}dx$$
$$=\frac{-1}{a}\int\frac{1-\cos^2 ax}{\cos ax}dx$$
$$=\frac{-1}{a}\int(\sec ax-\cos ax)dx$$
$$=\frac{-1}{a}\left[\frac{1}{a}\log(\sec ax+\tan ax)-\frac{\sin ax}{a}\right]$$
$$=\frac{-1}{a^2}\left[\log(\sec ax+\tan ax)-\sin ax\right]$$
$$=\frac{1}{a^2}\left[\sin ax-\log(\sec ax+\tan ax)\right]$$

$$\text{Q}=\int\frac{f_1X}{f_1f_2^{'}-f_1^{'}f_2}dx$$
$$=\int\frac{\cos ax\tan ax}{a}dx$$
$$=\frac{1}{a}\int\sin ax dx$$
$$=-\frac{1}{a^2}\cos ax$$

$\therefore P.I=Pf_1+Qf_2$

$$=\frac{1}{a^2}\cos ax[\sin ax-\log(\sec ax+\tan ax)]-\frac{1}{a^2}\sin ax[\cos ax]$$

$$= -\frac{1}{a^2}\left[\cos ax \log(\sec ax + \tan ax)\right]$$

$$y = C.F + P.I$$

$$= c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2}\left[\cos ax \log(\sec ax + \tan ax)\right]$$

7. **Solve** $\frac{d^2y}{dx^2} + y = \tan x$ **by the method of variation of parameters**.

Solution: The A.E is $m^2 + 1 = 0$

$$m = \pm i$$

C.F = $c_1 \cos x + c_2 \sin x$

Here $f_1 = \cos x, f_2 = \sin x$

$$f_1' = -\sin x, f_2' = \cos x$$

$$f_1 f_2' - f_2 f_1' = \cos^2 x + \sin^2 x = 1$$

$$P = -\int \frac{f_2 X}{f_1 f_2^1 - f_1' f_2} dx$$

$$= -\int \frac{\sin x \tan x}{1} dx$$

$$= -\int \frac{\sin^2 x}{\cos x} dx$$

$$= -\int \frac{1 - \cos^2 x}{\cos x} dx$$

$$= -\int (\sec x - \cos x) dx$$

$$= -\log(\sec x + \tan x) + \sin x$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_1' f_2} dx$$

$$= \int \frac{\cos x \tan x}{1} dx$$

$$= \int \sin dx$$

$$= -\cos x$$

$$\therefore P.I = Pf_1 + Qf_2$$

$$= -\cos x \log(\sec x + \tan x)$$

$$y = C.F + P.I$$

$$= c_1 \cos x + c_2 \sin x - \cos x \log(\sec x + \tan x)$$

8. **Solve by method of variation of parameters** $y'' - \frac{4}{x} y' + \frac{4}{x^2} y = x^2 + 1$

Solution: Given $y'' - \frac{4}{x} y' + \frac{4}{x^2} y = x^2 + 1$

i.e., $x^2 y'' - 4xy' + 4y = x^4 + x^2$

i.e., $\left[x^2 D^2 - 4xD + 4\right] y = x^4 + x^2$ …………(1)

Put x = e^z

$$\text{Logx} = \log e^{z}$$
$$= \text{z}$$
So that XD = D'
$$x^2 D = D'(D'-1)$$
$$(1) \Rightarrow [D'(D'-1) - 4D' + 4]y = (e^z)^4 + (e^z)^2$$
$$[D'^2 - 5D' + 4]y = e^{4z} + e^{2z}$$
A.E is $m^2 - 5m + 4 = 0$
$$(m-4)(m-1) = 0$$
$$m = 1,4$$
$$\therefore C.F = c_1 e^{4z} + c_2 e^{z}$$
Here $f_1 = e^{4z}, f_2 = e^{z}$
$$f_1' = e^{4z}, f_2' = e^{z}$$
$$f_1 f_2' - f_2 f_1' = e^{5z} - 4e^{5z} = -3e^{5z}$$
$$P.I = Pf_1 + Qf_2$$
$$\text{P} = -\int \frac{f_2 X}{f_1 f_2^1 - f_1' f_2} dx$$
$$= -\int \frac{e^z [e^{4z} + e^{2z}]}{-3e^{5z}} dx$$
$$= \int \frac{e^{5z} + e^{3z}}{3e^{5z}} dx$$
$$= \frac{1}{3} \int [1 + e^{-2z}] dz$$
$$= \frac{1}{3} \left[z + \frac{e^{-2z}}{-2} \right]$$
$$= \frac{1}{3} z - \frac{1}{6} e^{-2z}$$
$$\text{Q} = \int \frac{f_1 Z}{f_1 f_2' - f_1' f_2} dz$$
$$= \int \frac{e^{4z}(e^{4z} + e^{3z})}{-3e^{5z}} dz$$
$$= -\frac{1}{3} \int \frac{e^{8z} + e^{6z}}{e^{5z}} dz$$
$$= -\frac{1}{3} \int (e^{3z} + e^{z}) dz$$
$$= -\frac{1}{3} \left[\frac{e^{3z}}{3} + e^{z} \right]$$
$$= -\frac{1}{9} e^{3z} - \frac{1}{3} e^{z}$$
$$\therefore P.I = \left(\frac{1}{3} z - \frac{1}{6} e^{-2z} \right) e^{4z} + \left(-\frac{1}{9} e^{3z} - \frac{1}{3} e^{z} \right) e^{z}$$

$$=\frac{1}{3}ze^{4z}-\frac{1}{6}e^{2z}-\frac{1}{9}e^{4z}-\frac{1}{3}e^{2z}$$

$$y=C.F+P.I$$

$$=c_1e^{4z}+c_2e^{z}+\frac{1}{3}ze^{4z}-\frac{1}{6}e^{2z}-\frac{1}{9}e^{4z}-\frac{1}{3}e^{2z}$$

$$=c_1(e^{z})^4+c_2e^{z}+\frac{1}{3}z\left(e^{z}\right)^4-\frac{1}{6}\left(e^{z}\right)^2-\frac{1}{9}\left(e^{z}\right)^4-\frac{1}{3}\left(e^{z}\right)^2$$

$$=c_1\left(e^{z}\right)^4+c_2\left(e^{z}\right)+\frac{x^4}{3}\log x-\frac{x^4}{9}-\frac{x^2}{2}$$

DIFFERENTIAL EQUATIONS FOR THE VARIABLE COEFFICIENTS (CAUCHY'S HOMOGENEOUS LINEAR EQUATION)

Consider homogeneous linear differential equation as:

$$a^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{x-1}}+........+a_ny=X....................(1)$$

(Here a's are constants and X be a function of X) is called Cauchy's homogeneous linear equation.

$$a_nx^n\frac{d^ny}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+.............+a_1x\frac{dy}{dx}+a_0y=f(x)$$

is the homogeneous linear equation with variable coefficients. It is also known as Euler's equation.

Equation (1) can be transformed into a linear equation of constant Coefficients by the transformation.

$$x=e^{z},(or)z=\log x$$

Then

$$Dy=\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{1}{x}\frac{dy}{dz}$$

$$\left[D=\frac{d}{dx},\theta=\frac{d}{dz}\right]$$

$$x\frac{d}{dx}=xD=\frac{d}{dz}=\theta$$

Similarly

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$\frac{d^2y}{dx^2}=-\frac{1}{x^2}\frac{dy}{dz}+\frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$\therefore x^2\frac{d^2y}{dx^2}=\frac{d^2y}{dz^2}-\frac{dy}{dz}$$

$$\therefore x^2D^2 = (\theta^2 - \theta) = \theta(\theta - 1)$$

Similarly,

$$x^3D^3 = \theta(\theta-1)(\theta-2)$$

$$x^4D^4 = \theta(\theta-1)(\theta-2)(\theta-3)$$

and soon.

$\therefore xD = \theta$

$x^2D^2 = \theta(\theta-1)$

$x^3D^3 = \theta(\theta-1)(\theta-2)$

and so on.

1. Solve $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 4\sin(\log x)$

Solution: Consider the transformation

$$x = e^z, (or) z = \log x$$

$$\therefore xD = \theta$$

$$x^2D^2 = \theta(\theta-1)$$

$$(x^2D^2 + xD + 1)y = 4\sin(\log x)$$

$$(\theta^2 + 1)y = 4\sin(z)$$

R.H.S = 0 : $(\theta^2 + 1)y = 0$

A.E : $m^2 + 1 = 0, m = \pm i$

C.F = A cosz + B sinz

C.F = A cos (log x) +B sin (log x)

P.I = $\dfrac{1}{\theta^2 + 1}4(\sin z)$

$= 4\left(-\dfrac{z\cos z}{2}\right) = -2z\cos z$

P.I = $-2\log x\cos(\log x)$

$\therefore$ **Complete solution is:**

y = A cos (log x) +B sin (log x) $-2\log x\cos(\log x)$

y = $(A - 2\log x)\cos(\log x) - 2\log x\cos(\log x)$

2. Solve $x^2\dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} + 2y = x\log x$

Solution: Given $(x^2D^2 + 4xD + 2)y = x\log x$ **……………(1)**

Consider: $x = e^z, (or) z = \log x$

$$\therefore xD = \theta$$

$$x^2D^2 = \theta(\theta-1)$$

(1) : $(\theta(\theta-1) + 4\theta + 2)y = e^z z$

$$\left(\theta^2+3\theta+2\right)y = ze^z$$

A.E : $m^2+3m+2=0$

M = - 2, -1

C.F = $Ae^{-2z}+Be^{-z} = Ae^{\log x^{-2}}+Be^{\log x^{-1}}$

C.F = $\dfrac{A}{x^2}+\dfrac{B}{x}$

P.I = $\dfrac{1}{\left(\theta^2+3\theta+2\right)}\left(e^z z\right)$

$=\mathbf{e}^z \dfrac{1}{(\theta+1)^2+3(\theta+1)+2} z$

$= e^z \dfrac{1}{\left(\theta^2+5\theta+6\right)} z$

$= \dfrac{e^z}{6}\left(1+\left(\dfrac{\theta^2+5\theta}{6}\right)\right)^{-1} z$

$= \dfrac{e^z}{6}\left(1-\dfrac{5}{6}\theta\right)z = \dfrac{e^z}{6}\left(z-\dfrac{5}{6}\right)$

$= \dfrac{e^{\log x}}{6}\left(\log x-\dfrac{5}{6}\right)$

$= \dfrac{x}{6}\left(\log x-\dfrac{5}{6}\right)$

Complete solution is y = C.F +P.I

$$= \frac{A}{x^2}+\frac{B}{x}+\frac{x}{6}\left(\log x-\frac{5}{6}\right)$$

3. Solve ($(x^2D^2-3xD+4)y = x^2, given that$ **y(1) = 1 and y'(1) = 0**

Solution: Given $\left(x^2D^2-3xD+4\right)y = x^2$ **…………………(1)**

Take $x=e^z, (or) z=\log x$

$\therefore xD=\theta$

$x^2D^2=\theta(\theta-1)$

(1) : $\left(\theta^2-4\theta+4\right)y = e^{2z}$

A.E : $m^2-4m+4=0, m=2,2$

C.F = $(A+Bz)e^{2z} = (A+B\log x)x^2$

P.I = $\dfrac{1}{(\theta-2)^2}\left(e^{2z}\right) = e^{2z}\dfrac{1}{(\theta+2-2)^2}(1)$

$= e^{2z}\dfrac{1}{\theta^2}(1)$

P.I = $e^{2z}\dfrac{z^2}{2}$

$= \dfrac{x^2(\log x)^2}{2}$ **…………………(2)**

Complete solution is : y = $(A + B\log x)x^2 + \dfrac{x^2(\log x)^2}{2}$

Apply conditions: y(1) =1,y'(1) = 0 in (2)

A = 1, B = -2

Complete solution is y = $(1 - 2\log x)x^2 + \dfrac{x^2(\log x)^2}{2}$

EQUATION REDUCIBLE TO THE HOMOGENEOUS LINEAR FORM (LEGENDRE LINEAR EQUATION)

It is of the form:

$$(a+bx)^n \frac{d^n y}{dx^n} + A_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \text{..........} + A_n y = f(x)$$ **…………..(1)**

$A_1, A_2, \ldots\ldots, A_n$ **are some constants**

It can be reduced to linear differential equation with constant Coefficients,

by taking: $a + bx = e^z\,(or) z = \log(a+bx)$

Consider $\dfrac{d}{dx} = D, \dfrac{d}{dz} = \theta, gives$

$(a+bx)\dfrac{dy}{dx} = b\dfrac{dy}{dz} \Rightarrow (a+bx)Dy = b\theta(y)$

Similarly $(a+bx)^2 D^2 y = b^2\theta(\theta-1)y$ **…………(2)**

$(a+bx)^3 D^3 y = b^3\theta(\theta-1)(\theta-2)$ **and so on**

Substitute (2) in (1) gives: the linear differential equation of constant Coefficients.

Solve $(2x+3)^2 y'' - (2x+3)y' - 12y = 6x$

Solution: This is Legendre's linear equation:

$(2x+3)^2 y'' - (2x+3)y' - 12y = 6x$ **……………….(1)**

Put z = log (2x + 3) , $e^z = 2x+3$

$(2x+3)D = 2\theta$

$(2x+3)^2 D^2 = 4(\theta^2 - \theta), \theta = \dfrac{d}{dz}$

Put in (1) : $(4\theta^2 - 6\theta - 12)y = 3e^z - 9$

R.H.S =0

A.E : $4m^2 - 6m - 12 = 0$

$m_1 = \dfrac{3+\sqrt{57}}{4}, m_2 = \dfrac{3-\sqrt{57}}{4}$

C.F = $Ae^{m_1 z} + Be^{m_2 z}$

C.F = $A(4x+3)^{m_1} + B(2x+3)^{m_2}$

$$\textbf{P.I}_1 = \frac{3e^{z}}{4\theta^2 - 6\theta - 12} = -\frac{3}{14}(2x+3)$$

$$\textbf{P.I}_2 = \frac{9e^{\theta z}}{4\theta^2 - 6\theta - 12}$$

$$= -\frac{9}{12} = -\frac{3}{4}$$

Solution is

$$\textbf{y} = A(2x+3)^{\left(3+\sqrt{57/4}\right)} + B(2x+3)^{\left(3-\sqrt{57}/4\right)} - \frac{3}{14}(2x+3) - \frac{3}{4}$$

SIMULTANEOUS FIRST ORDER LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

1. Solve the simultaneous equations, $\frac{dx}{dt} + 2x + 3y = 2e^{2t}, \frac{dy}{dt} + 3x + 2y = 0$

Solution: Given $\frac{dx}{dt} + 2x + 3y = 2e^{2t}, \frac{dy}{dt} + 3x + 2y = 0$

Using the operator D = $\frac{d}{dt}$

$(D+2)x + 3y = 2e^{2t}$ **………………(1)**

$3x + (D+2)y = 0$ **……………….(2)**

Solving (1) and (2) eliminate (x) :

$3\times(1) - (2)\times(D+2) \Rightarrow (D^2 + 4D - 5)y = -6e^{2t}$ ....(3)

A.E : $m^2 + 4m - 5 = 0$

m = 1,-5

C.F = $Ae^{t} + Be^{-5t}$

P.I = $\frac{-6e^{2t}}{D^2 + 4D - 5} = -\frac{6}{7}e^{2t}$

y = $Ae^{t} + Be^{-5t} - \frac{6}{7}e^{2t}$

put in (1) : x = $-\frac{1}{3}[(D+2)y]$

x = $-Ae^{t} + Be^{-5t} + \frac{8}{7}e^{2t}$

$\therefore$ *solution* **is :**

x = $-Ae^{t} + Be^{-5t} + \frac{8}{7}e^{2t}$

y = $Ae^{t} + Be^{-5t} - \frac{6}{7}e^{2t}$

2. Solve $\frac{dx}{dt} + y = \sin t, \frac{dy}{dt} + x = \cos t, giventhat$ **t=0, x = 1, y =0**

Solution: Dx + y = sin t ……………..(1)

x + Dy = cost ……………… (2)

Eliminate x : (1) – (2)D $\Rightarrow y - D^2 y = \sin t + \sin t$

$$\left(D^2-1\right)y=-2\sin t \text{..............}(3)$$

$$m^2-1=0, m=\pm 1$$

C.F = Ae$^{t}+Be^{-t}$

P.I = $-2\dfrac{\sin t}{D^2-1}=(-2)\dfrac{\sin t}{-1-1}=\sin t$

y = **Ae**$^{t}+Be^{-t}$ **+ sint**

(2) : x = cost –D(y)

x = cost - $\dfrac{d}{dt}\left(Ae^{t}+Be^{-t}+\sin t\right)$

x = $\cos t-Ae^{t}+Be^{-t}-\cos t$

x = $-Ae^{t}+Be^{-t}$

Now using the conditions given, we can find A and B

$$t=0, x=1\Rightarrow 1=-A+B$$

$$t=0, y=0\Rightarrow 0=A+B$$

B = $\dfrac{1}{2}$**, A =** $-\dfrac{1}{2}$

Solution is

x = $\dfrac{1}{2}e^{t}+\dfrac{1}{2}e^{-t}=\cosh t$

y = $-\dfrac{1}{2}e^{t}+\dfrac{1}{2}e^{-t}+\sin t=\sin t-\sinh t$

3. Solve $\dfrac{dx}{dt}+2y=-\sin t, \dfrac{dy}{dt}-2x=\cos t$

Solution: Dx + 2y = -sin t ……………..(1)

- 2x +Dy = cos t ……………...(2)

(1) $\times 2+(2)\times D\Rightarrow 4y+D^2y=-2\sin t+D(\cos t)$

$$\Rightarrow \left(D^2+4\right)=-3\sin t$$

$$m^2+4=0, m=\pm i2$$

C.F = $A\cos 2t+B\sin 2t$

P.I = $-\dfrac{3\sin t}{D^2+4}=\dfrac{-3\sin t}{-1+4}=-\sin t$

y = $A\cos 2t+B\sin 2t$ **- sin t**

(2) : x = $\dfrac{1}{2}\left[Dy-\cos t\right]$

x = $\dfrac{1}{2}\left[\dfrac{d}{dt}\left(A\cos 2t+B\sin 2t-\sin t\right)-\cos t\right]$

x = A cos2t +Bsin2t – cost

Solution is :

x = A cos2t +Bsin2t – cost

y = $A\cos 2t+B\sin 2t$ **- sin t**

4. Solve $\frac{dx}{dt}-\frac{dy}{dt}+2y=\cos 2t, \frac{dx}{dt}+\frac{dy}{dt}-2x=\sin 2t$

Solution: Dx + (-D +2)y = cos 2t …………..(1)

(D – 2)x +Dy = sin 2t ……………(2)

Eliminating y from (1) and (2)

$$\left(D^2-2D+2\right)x=-2\sin 2t+\cos 2t$$

R.H.S = 0 $\Rightarrow m^2-2m+2=0$

m = $1\pm i$

C.F = $e^t\left(A\cos t+B\sin t\right)$

P.I $_1$ **=** $\frac{\left(-2\sin 2t\right)}{D^2-2D+2}=\frac{\sin 2t}{1+D}$

$=\frac{\left(1-D\right)}{1-D^2}\sin 2t=\frac{\sin 2t-D(\sin 2t)}{1+4}$

$=\frac{\sin 2t-2\cos 2t}{5}$

P.I $_2$ **=** $\frac{1}{D^2-2D+2}\left(\cos 2t\right)$

$=-\frac{\left(\cos 2t+2\sin 2t\right)}{10}$

x = $e^t(A\cos t+B\sin t)+\frac{\sin 2t-2\cos 2t}{5}-\frac{\left(\cos 2t+2\sin 2t\right)}{10}$

(1) +(2) $\Rightarrow 2\frac{dx}{dt}+2y-2x=\cos 2t+\sin 2t$

2y = cos2t +sin 2t + 2x - $2\frac{dx}{dt}$

y = $\frac{1}{2}\left[\cos 2t+\sin 2t+2x-2\frac{dx}{dt}\right]$ **…………(3)**

Substitute x in (3)

y = $e^t\left(A\cos t-B\sin t\right)-\frac{\sin 2t}{2}$

Solution is :

x = $e^t(A\cos t+B\sin t)+\frac{\sin 2t-2\cos 2t}{5}-\frac{\left(\cos 2t+2\sin 2t\right)}{10}$

y = $e^t\left(A\cos t-B\sin t\right)-\frac{\sin 2t}{2}$